The 1st Annual West Windsor-Plainsboro Mathematics Expo

Saturday, October 26$^{\text{th}}$, 2019

Grade 8 Problem Set

Directions:

Solve the following problems to the best of your ability. If you do not understand a problem or cannot solve it, skip it or ask for a hint. If you cannot solve a problem even after receiving all the hints for that problem, wait until the 30 minute mark and ask a proctor for further help or the solution. Some problems may not have hints.

Calculators are not allowed for these problems. You may, however, discuss with the people around you after 30 minutes have passed. That being said, do not ruin a problem for somebody by giving them a solution before they have a chance to attempt the problem themselves.

For this test, there will be 20 questions, and you will have a time limit of 60 minutes in total, which will be split into 30 minutes of individual work and 30 minutes of collaborative work. This test is very long and you are not expected to be able to do all of the problems. We recommend picking a range of 10-15 problems to work on.

Please note that this is not a competition, and your goal is to enjoy the problems and gain experience.

HAVE FUN!

By the way, if you finish this exceptionally early, you are most likely an exceptional student. Thus, here is a slightly harder problem that you may wish to solve:

CHALLENGE:

$$\left(\lim_{x\to 0}\frac{x-\sin x}{x^3}\right)^{-1}$$

1. Evaluate: $2 + 2 \times 2$

Solution: 6

Hint: Seriously?

2. Define $A(a, b)$

$$A(a,b) = \begin{cases} \text{b+1} & \text{if } a = 0 \\ \text{A(a-1, 1)} & \text{if } a > 0 \text{ and } b = 0 \\ \text{A(a-1, A(a, b-1))} & \text{if } a > 0 \text{ and } b > 0 \end{cases}$$

a) Find $A(0, 2)$

b) Find $A(1, 2)$

c) Find $A(2, 2)$

d) Find $A(3, 2)$

Solution:

a) 3

b) 4

c) 7

d) 29

Hint: Try to keep simplifying until you're done. Keep plugging in values until you have the answer, and try to keep the work neat and tidy. The sequence will eventually end.

3. Suraj has a lot of debt. He has worked out a payment plan where he pays \$5 every day in the morning, but at the end of each day if he still has debt, \$2 of interest will be added. If he currently has \$25 of debt and the day has just started (has not paid morning payment), how many days will it take to get rid of all the debt?

Solution: Day 1: 22

Day 2: 19

Day 3: 16

Day 4: 13

Day 5: 10

Day 6: 7

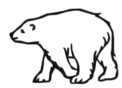
Day 7: 4

Day $\boxed{8}$: 0

Hint: Write what happens each day out.

4. David the bear loves to break down wire fences that are used to keep bears out. He has two wires of the same length. He creates an equilateral triangle from the first one, and he creates a square from the second one. Find the ratio of the area of these 2 shapes. (triangle to square)

> **Solution:**
>
> If the length is $12s$, the area of triangle is $\frac{16s^2\sqrt{3}}{4}$ and the square is $9s^2$ so the ratio is $\boxed{\frac{4\sqrt{3}}{9}}$
>
> **Hint:** Let the length be l. If they don't know the area of an equilateral triangle tell them to drop a height and use Pythagorean.



5. Five cows are in a line. Farmer John wants to paint each of them 1 of 3 colors, but if two cows next to each other are both the same color he gets confused. Find the number of ways he can paint his cows without getting confused.

> **Solution:** First cow has 3 possible colours, second cow has 2 (not the first cow), third cow has 2, fourth cow has 2, fifth cow has 2. $3 \times 2 \times 2 \times 2 \times 2 = \boxed{48}$
>
> **Hint:** For each cow, how many colors can it be painted?

6. a) Let m and n be positive integers such that $3m + 2n = 2019$. How many values of m and n are there?

   b) Let m and n be positive integers such that $3m + 2n \leq 2019$. How many values of m and n are there?

> **Solution:** a) m = odd numbers from 1 to 671, which is $\boxed{336}$
>
> b) Case work down. Doing the same thing as before for 2017 and 2016 and so on will give
>
> $6 \times (335 + 334 + \cdots + 1) + 4 \times 336$ or $\boxed{339024}$.
>
> **Hint:** a) Find a value that works, then try to change one of the variables.
>
> b) Let $3m + 2n$ equal 2019, then 2018 and so on.

7. Kagami takes 1 hour to walk from her house to her school. She takes the bus on the way home which only takes 20 minutes. Given that her average speed is 9 km/hr, how far away is the school in km?

> **Solution:** $\frac{d}{\frac{1}{1}+\frac{1}{3}} = 9$ $\boxed{d = 6\text{km}}$
>
> **Hint:** Let the distance be d, set up equation, solve. Use d = rt.

8. Adi sucks at handling money. He currently has \$20. Someone went up to him and said, "I will increase your money but 10%, but then I would take 10% away from you". Adi agreed, what is the difference between amount of money he had before and the amount of money he has now?

Solution: $\$20 \times \frac{11}{10} \times \frac{9}{10} = \boxed{\$0.20}$

Hint: Do it step by step, what happens first, what do you have, then what happens second.

9. Inazuma and Ikazuchi both went to an ice-cream stop on a certain day between 2:00 pm and 3:00 pm, staying for a total of 10 minutes each. What is the probability that they were able to meet up?

Solution:

Graphing it out gives a square with a strip across the middle. Finding the area of the strip gives the probability to be $\boxed{\frac{11}{36}}$

Hint: Graph it out. Let the x be times Inazuma arrives and y be the time Ikazuchi arrives. When do they meet? Try to draw out an area.

10. Circle O_0 and circle O_1 intersect at A and B. Given that the radius of circle O_0 is 24 and the radius of circle O_1 is $24\sqrt{3}$, and $\overline{AB}$ is $24\sqrt{3}$, find the area of the intersection of the circles.

Solution:

$\frac{24^2\pi}{3} - 12^2\sqrt{3} + \frac{24^2 \times 3\pi}{6} - \frac{24^2 \times 3\sqrt{3}}{4} = \boxed{480\pi - 576\sqrt{3}}$

Hint: Break the area up into smaller sections Draw lines to connect triangles with known area and sections of a circle with known area. Be careful when calculating.

11. Let $N = 2019^{10}$. How many factors of N^2 less than N are there that do not divide N.

Solution:

$N^2 = 3^{20} \times 673^{20}$ has 441 factors, 220 of which are less than N, since they come in pairs of larger and less than N. Since N has 120 factors less than N (221 - 1 from itself), the answer is $220 - 120 = \boxed{100}$.

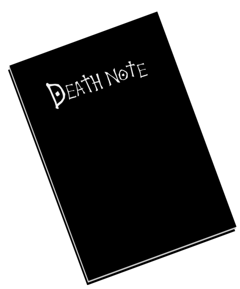
Hint: All factors of N^2 must come in pairs. What is special about the pairs and why is that relevant to N?

12. Violet is randomly typing keys on her typewriter. There are 45 keys, 25 of which are letters and the rest are punctuation. She randomly types out 7 characters. What is the probability that she just typed 2 different 3-lettered "words" (only letters) with a punctuation in between. For example, "AAA.BBB" is one such possibility.

> **Solution:** The answer is $(\frac{25}{45})^6 \times \frac{20}{45} = \boxed{\frac{62500}{4782969}}$.
>
> **Hint:** Should be an easy break from the harder questions surrounding it.

13. Pratyoy Biswas is forced to watch Death Note! Given that each day there is a 50% chance he is too busy, 25% chance he watches an episode, and a 25% chance he watches two episodes(if only one episode remains he only watches one episode). If Death Note is a 5 episode show, what is the expected number of days to watch the entire thing?

> **Solution:**
>
> Let e_i be the expected number of days left if there is i episodes left. e_0 is obviously 0 and $e_1 = \frac{e_0}{2} + \frac{e_1}{2} + 1 = 2$. We also have the equation $e_i = \frac{e_{i-1}}{4} + \frac{e_{i-2}}{4} + \frac{e_i}{2} + 1$ for all i 1, which gives a system of equations. The desired number is e_5, which after solving the system of equations gives $\boxed{e_5 = \frac{57}{8}}$.
>
> **Hint:** Make variables that represent the expected number of days to finish it if there is 1 episode left, 2 episodes, 3 episodes... Find the relationship between these variables.



14. Let a, b, c be the roots of the polynomial $x^3 - 5x^2 + 9x - 12$. Find $a^2 + b^2 + c^2$.

> **Solution:**
>
> Using Vieta's we can see that $a + b + c = 5$ and $ab + bc + ac = 9$. Squaring $a + b + c$ gives $a^2 + b^2 + c^2 + 2(ab + bc + ac) = 25$, meaning $\boxed{a^2 + b^2 + c^2 = 7}$.
>
> **Hint:** Square the sum. If they don't know Vieta's tell them to expand $(x - a)(x - b)(x - c)$

15. Verniy is shooting targets. She has to shoot 3 strings of 3 targets. For each string, the targets are arranged vertically such that hitting the top target will make the whole column fall and hitting the middle would make the middle and bottom targets fall. How many ways can she shoot at the targets such that no target remains after she is done? Verniy is amazing at shooting and does not miss. The order in which Verniy takes the shots matters.

Solution:

This question can be simplified into a 7 part case-work. Give that you must hit the lowest target first is the equivalent of ordering 3 A's, 3 B's and 3 C's in a string of 9 characters. Then using casework on the amount of times you hit each target (once, twice or three times) gives:

(Hit all 9) $\binom{9}{3}\binom{6}{3}+$

(Hit 8) $\binom{3}{1}\binom{8}{3}\binom{5}{3}\times 2+$

(Hit 7) $\binom{3}{2}\binom{7}{3}\binom{4}{2}\times 4+\binom{3}{1}\binom{7}{6}\binom{6}{3}+$

(Hit 6) $\binom{6}{2}\binom{4}{2}\times 8+\binom{3}{1}\binom{2}{1}\binom{6}{1}\binom{5}{3}\times 2+$

(Hit 5) $\binom{3}{1}\binom{5}{1}\binom{4}{2}\times 4+\binom{3}{2}\binom{5}{1}\binom{4}{1}+$

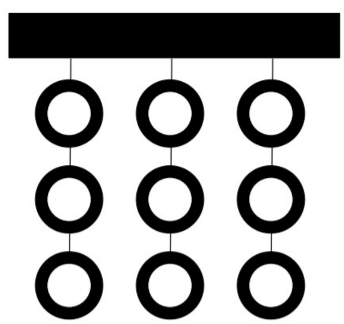
(Hit 4) $\binom{3}{1}\binom{4}{1}\binom{3}{1}\times 2+$

(Hit 3) $\binom{3}{1}\binom{2}{1}$

$=\boxed{9918}$

(There are several $\times 2$ or $\times 4$ since whenever you only shoot 2 targets in a column you can either shoot bottom top or middle top)

Hint: Consider each string as A, B, C. Assume you can only hit the bottom one first, tell them to solve. Then tell them it's just casework.



16. How many non-empty subsets S of the set $\{1, 2, 3, \ldots, 15\}$ have the following two properties?

(1) No two consecutive integers belong to S.

(2) If S contains k elements, then S contains no number less than k.

Solution: This question can be solved fairly directly by casework and pattern-finding. We give a somewhat more general attack, based on the solution to the following problem:

How many ways are there to choose k elements from an ordered n element set without choosing two consecutive members?

You want to choose k numbers out of n with no consecutive numbers. For each configuration, we can subtract $i-1$ from the i-th element in your subset. This converts your configuration into a configuration with k elements where the largest possible element is $n-k+1$, with no restriction on consecutive numbers. Since this process is easily reversible, we have a bijection. Without consideration of the second condition, we have: $\binom{15}{1}+\binom{14}{2}+\binom{13}{3}+...+\binom{9}{7}+\binom{8}{8}$

Now we examine the second condition. It simply states that no element in our original configuration (and hence also the modified configuration, since we don't move the smallest element) can be less than k, which translates to subtracting $k-1$ from the "top" of each binomial coefficient. Now we have, after we cancel all the terms $\binom{n}{k}$ where $n<k$, $\binom{15}{1}+\binom{13}{2}+\binom{11}{3}+\binom{9}{4}+\binom{7}{5}=15+78+165+126+21=\boxed{405}$

Hint: For five elements, if the elements are a_0, a_1, a_2, a_3, a_4 consider $a_0, a_1-1, a_2-2, a_3-3, a_4-4$. What is special about those 5 values. If they still can't figure it out tell them to write some examples of things that work and things that don't. Expand to other sizes.

17. a) An ant is currently at $(0,0)$ and is trying to get to its base at $(5,5)$. There is currently a wall that is the line $y=x$. If it can not cross this wall, but can be on it, how many ways can it get to $(5,5)$ given that it can only move one unit right or one unit up at a time.

b) How many ways are there to write out 5 pairs of parentheses such that it is a valid combination of parentheses? Ex. (((())())) or (()(())()) not (())())(() or))((()()(

Solution:

a) One way to do this is to draw the grid out and label the points a number that corresponds to the number of ways to reach that point. First assume the ant moves up as it is symmetric. The number for points (x,y) is $(x-1,y)+(x,y-1)$ given that it is not on the wall. If it is on the wall then it is just $(x-1,y)$. Adding all the points gives 42 so the total is just $\boxed{84}$.

b) Consider each '(' as a movement up and each ')' as a movement across. This problem becomes the problem before, so the answer is $\boxed{42}$

Hint: a) Label the points with the number of ways to get to that point from the centre. Find the relationship between points.

b) Why is the problem split into two parts, how is this related to the previous problem?

18. There is a naval base that is 12 docks side by side. An aircraft carrier would require 3 docks, a cruiser would require 2 and a small submarine would require 1. Given an infinite amount of all types of ships, and all ships of the same class are indistinguishable, how many ways are there to fill all the docks of the naval base?

Solution:

This is a simple recursion. Making a chart of the number of ways to fill a base with n docks, one can see that the number of ways to fill a dock of size a is just the ways to fill $a-1$ + the number of ways to fill $a-2$ slots + the number of ways to fill $a-3$ slots (since you could simply add on a small sub, cruiser, or aircraft carrier respectively to reach n filled docks), which gives a sequence similar to the Fibonacci sequence, $a_n = a_{n-1}+a_{n-2}+a_{n-3}$. Solving this gives the answer of $\boxed{927}$

Hint: Consider smaller problems and expand outwards. What if the base had 1 dock, 2 docks...

19. How many decreasing sequences $a_1, a_2, ..., a_{2019}$ of positive integers are there such that $a_1 \leq 2019^2$ and $a_n + n$ is even for each $1 \leq n \leq 2019$?

Solution:

We can add a $a_0 = 0$ and a $a_{2020} = 2019^2 + 1$. The only condition for the sequence is that they alternate between parities, so if we consider all the differences between a values, they turn out to be odd numbers that sum to $2019^2 + 1$. We can represent the differences between a_i and a_{i-1} as $2c_i - 1$ so the sum of c_i is $\binom{2020}{2} + 1$. This is basically placing 2019 dividers between $\binom{2020}{2} + 1$ so the number of sequences is $\boxed{\binom{\binom{2019}{2}}{2019}}$

Hint: Add a $a_0 = 0$ and a $a_{2020} = 2019^2 + 1$. Don't calculate the entire thing out please.

20. Let a, b be positive real numbers with $a > b$. Compute the minimum possible value of the expression $\frac{a^2b - ab^2 + 8}{ab - b^2}$

Solution:

This can be expressed as $a + \frac{8}{b(a-b)}$ Using AM-GM, one can see that the largest possible value for $b(a-b)$ is $\frac{a^2}{4}$ so the expression is just $a + \frac{32}{a^2}$. We can use AM-GM again. $a + \frac{32}{a^2} = \frac{a+a+\frac{64}{a^2}}{2} \geq \frac{3}{2}(a \times a \times \frac{64}{a^2})^{\frac{1}{3}}$ giving a minimum value of 6 which is possible when $a = 4$ and $b = 2$.

Hint: AM-GM. Try to simplify the expression until you're stuck, then try to find a minimum value through AM-GM.