The 1st Annual West Windsor-Plainsboro Mathematics Expo

Saturday, October 26$^{\text{th}}$, 2019

Grade 7 Problem Set

Directions:

Solve the following problems to the best of your ability. If you do not understand a problem or cannot solve it, skip it or ask for a hint. If you cannot solve a problem even after receiving all the hints for that problem, wait until the 30 minute mark and ask a proctor for further help or the solution. Some problems may not have hints.

Calculators are not allowed for these problems. You may, however, discuss with the people around you after 30 minutes have passed. That being said, do not ruin a problem for somebody by giving them a solution before they have a chance to attempt the problem themselves.

For this test, there will be 20 questions, and you will have a time limit of 60 minutes in total, which will be split into 30 minutes of individual work and 30 minutes of collaborative work. This test is very long and you are not expected to be able to do all of the problems. We recommend picking a range of 10-15 problems to work on. Please note that this is not a competition, and your goal is to enjoy the problems and gain experience.

HAVE FUN!

By the way, if you finish this exceptionally early, you are most likely an exceptional student. Thus, here is a slightly harder problem that you may wish to solve:

CHALLENGE:

Let a and b be positive integers such that $(1+ab)|(a^2+b^2)$. Show that $\frac{(a^2+b^2)}{(1+ab)}$ must be a perfect square.

If you think it is too hard, don't worry, because it is the toughest problem on IMO :)

1. Evaluate $(2^{(2^2)})) - (1^{(1^1)})$

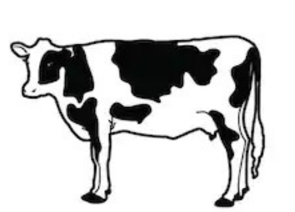
> **Solution:** $(2^{(2^2)})) - (1^{(1^1)}) = 2^4 - 1 = 16 - 1 = 15$:)
>
> **Hint:** Just calculate the values

2. Find the probability of getting a sum of 8 after rolling two fair 6-sided dice.

> **Solution:** Just list all the possibilities: (2, 6), (3, 5), (4, 4), (5, 3), (6, 2) So yeah, the answer is $\frac{5}{36}$:)
>
> **Hint:** Just do it

3. One day, Bessie the cow decides to wander in her farm. First, she walks west for 15 miles, north 12 miles, and east 2 miles. Find the distance between her endpoint and starting point.

> **Solution:** Since Bessie walked 15 miles west and 2 miles east, that means she walked 13 miles west. Using Pythagorean Theorem, we get the distance is $\sqrt{13^2 + 12^3} = \sqrt{313}$:)
>
> **Hint:** Pythagorean Theorem



4. Find the unit digit of $2019^{2019} - 2019!$.

> **Solution:** We can first find the unit digits of the two terms separately. The unit digit of 2019! is 0. The unit digit of 2019^{2019} is 9. Since $2019^{2019} < 2019!$, the final answer is negative, which means that it has a units digit of 9.
>
> **Hint:** Find some pattern using small terms

5. Find the last 2 digits of $2019! - 25^{2019}$.

> **Solution:** The last 2 digits of the first term is 0. The last 2 digits of the second term is 25. Therefore, the last 2 digits is 75. :)
>
> **Hint:** Find some pattern using small terms

6. What is the area of hexagon $\overline{ABCDEF}$ with A=(1, 1), B=(4, 2), C=(9, 3), D=(16, 4), E=(25, 5), F=(36, 6).

> **Solution:** Just a shoelace bash. (If you don't know shoelace, just search it on Google) After applying the formula, we get 20 is the area :)
>
> **Hint:** Use shoelace theorem

7. Let $f(x) = 1 + \frac{1}{x}$. Find $\prod_{i=1}^{2019} f(i)$. Note: This is simply asking you to find $f(1) \cdot f(2) \cdot f(3) \ldots \cdot f(2019)$.

> **Solution:** We can rewrite $f(x)$ as $\frac{1+x}{x}$ $f(1) = 2$, $f(1)(2) = 3$, $f(3)(2)(1) = 4$... Therefore, $\prod_{i=1}^{2019} f(i) =$ 2020 :)
>
> **Hint:** Find some pattern by testing out the product up to f(1), f(2), f(3), and so on.

8. One day, Bessie the cow encountered 2020 rooms. In each room, there is one light bulb, and all the light bulbs are initially off. Since Farmer John is not supervising her, she decides to play around with the rooms. Specifically, she wants to turn on some of the light bulbs. She visits all the rooms, and she has $\frac{1}{2}$ of chance to turn on each light bulb. But she is very afraid of darkness, so at the end of the process, it is given at least one light bulb that is turned on. Find the expected number of light bulbs that is on at the end of the process.

> **Solution:** Assuming that it is possible that no light bulb is turned on, then it is easy to see that the expected value is 1010. Therefore, out of all the 2^{2020} cases, the sum of the number light bulbs that are turned on is $1010 \cdot 2^{2020}$. Now, we can subtract the case where no light bulb is on. Since it doesn't change the sum, so the answer is $\frac{1010 \cdot 2^{2020}}{2^{2020}-1}$:)
>
> **Hint:** There is only one case that doesn't work

9. Let f(x) be a monic 4th degree polynomial such that f(1) = 2, f(2) = 5, f(3) = 10, f(4) = 17. Find f(5).

> **Solution:** After observation, we have $f(x) - x^2 - 1$ has 1, 2, 3, 4 as the roots. Since $f(x)$ is a 4th degree monic polynomial, $f(x) = (x-1)(x-2)(x-3)(x-4) + x^2 + 1$, therefore, $f(5) = 50$:)
>
> **Hint:** $2 = 1^2 + 1, 5 = 2^2 + 1...$

10. Find the number of ways to arrange 3 reds identical beads, 3 blue identical beads, and 3 green identical beads on a necklace. Given that 2 arrangements are considered different if and only if one cannot obtain the other by rotation or reflection.

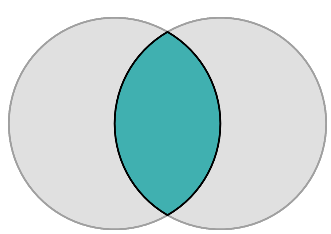
> **Solution:** We can apply Burnside's Lemma to this problem. We can see that the identity is $\binom{9}{3}\binom{6}{3}$, 120 and 240 degree rotation have 6 ways, and rest of the transformation has 0 ways. Therefore, the answer is $\frac{\binom{9}{3}\binom{6}{3}+6+6}{18} = 94$:)
>
> **Hint:** Burnside's Lemma

11. Let $t(x)$ denotes the number of divisors of x. Find the sum of a less than 60 such that $t(a) - t(a-1)$ is a positive odd integer.

> **Solution:** Since $t(a) - t(a-1)$ is odd, therefore, either a or $a-1$ is a perfect square. We can check all the cases, we get that 2, 4, 10, 16, 26, 36, 50 works, so the answer is 144. :)
>
> **Hint:** Think about perfect squares

12. Let w_1 and w_2 be two circles that intersect at P and Q. Let A, B be on w_1, and C, D be on w_2 such that $\overline{AC}$ passes through P and $\overline{BD}$ passes through P. Given $\overline{QA} = 4$, $\overline{QC} = 6$, $\overline{QB} = 2$, find $\overline{QD}$.

> **Solution:** Using spiral similarity, we can conclude that $\triangle QAC$ is similar to $\triangle QBD$, therefore, $\overline{QD} = 3$:)
>
> **Hint:** Similar Triangle



13. Let a_n and b_n be a sequence of real numbers, and $a_0 = 1$ and $b_0 = 1$. In addition for $k > 0$, $a_k = a_{k-1}^2$-b_{k-1}^2, and $b_k = 2a_{k-1}b_{k-1}$. Find $(a_{2019})^2 + (b_{2019})^2$.

> **Solution:** Base on the recurrence, we can get a new recurrence: $a_{k+1} + b_{k+1}i = (a_k + b_k i)^2$, therefore, $a_{2019} + b_{2019}i = (a_0 + b_0 i)^{2^{2019}}$. Since the magnitude of $a_0 + b_0 i$ is $\sqrt{2}$, therefore, the square of the magnitude $a_{2019} + b_{2019}i$ is 2^{2020}. :)
>
> **Hint:** Use Complex Number

14. Find the first positive integer n such that $3^n \equiv 2019^n \pmod{343}$.

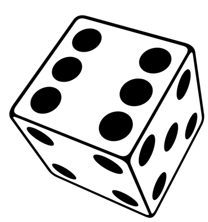
> **Solution:** We use Lifting the Exponent technique for this problem. $v_7(2019^n - 3^n) = v_7(2016) + v_7(n) = 1 + v_7(n) \geq 3$ Therefore, the minimum n that satisfy the inequality is 49. :)
>
> **Hint:** Use Lifting the Exponent

15. Let P be a point in △ ABC. Given that AB=AC, ∠ BAC = 102°, ∠ PBA=9°, ∠ PAB=21°. Find the degree of ∠ APC.

> **Solution:** Construct a equilateral triangle APD such that $\angle DAC = 21°$, therefore, $\triangle PDC \cong \triangle ADC$. Therefore, $\angle APC = 69°$:)
>
> **Hint:** Construct an equilateral triangle

16. Tossing dice has always been Elsie's favorite activity. She is so addicted that she wouldn't give up until she tosses all numbers from 1 through 3 consecutively in that order at least once. Find the probability that Elsie finishes tossing her dice using exactly 9 tosses?

> **Solution:** Let a_n be the probability that Elsie tosses exactly n times. Therefore, $a_0 = a_1 = a_2 = 0$. We can also build a recursion, $a_k = \frac{1}{27} \cdot (1 - a_{k-3})$ for $k \geq 3$ Therefore, $a_9 = \frac{703}{19683}$:)
>
> **Hint:** Use recursion



17. Find the following sum: $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m}{3^{n+m} \cdot (m+n)}$

> **Solution:** Let the desire sum be S. Since m and n are symmetrical, therefore $2S = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m+n}{3^{n+m} \cdot (m+n)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{3^{n+m}} = \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot \sum_{m=1}^{\infty} \frac{1}{3^m} = \frac{1}{4}$:)
>
> **Hint:** Make it symmetrical

18. Find the maximum number of different values that can be obtained from the expression a/b/c/d/e/f using 5 pairs of parentheses for some real non-zero a, b, c, d, e, f. Every pair of parentheses must be non-trivial. i. e, "(a)" or "((a/b))" are not allowed.

Solution: Well actually, this is a catalan number formation. Since division is non associative, therefore, the answer would just be $C_5 = 42$:)

Hint: Catalan numbers

19. Find the minimum real value of $\sqrt{x^2 + 20x + 1000} + \sqrt{x^2 - 8x + 272}$

Solution: We can rewrite $\sqrt{x^2 + 20x + 1000} + \sqrt{x^2 - 8x + 272}$ as $\sqrt{(x + 10)^2 + 900} + \sqrt{(4 - x)^2 + 256}$ We can think of this as the distance from (0, 0) to (x+10, 30), and the distance from (x+10, 30) to (14, 46). Therefore, the minimum sum is $\sqrt{14^2 + 46^2} = 2\sqrt{578}$:)

Hint: Think about coordinates and distances

20. Let w_1 and w_2 be two circles, and let l be the external common tangent line which tangents to w_1 and w_2 at A and B, respectively. Let P and Q be the two intersections of the circles, and P is closer to l than Q. Let $\overline{AC}$ be parallel to $\overline{BQ}$ and C is on w_1. Let $\overline{BD}$ be parallel to $\overline{AQ}$ and D is on w_2. Given that ∠PAB = 15°, and ∠PBA = 30°. Find $\overline{CP}/\overline{DP}$.

Solution: The answer is $2+\sqrt{3}$ This problem is pretty difficult, and the solution is quite long. It involves spiral similarity and trig bash. So I won't provide it. :(

Hint: It's too hard, just give up