The 1st Annual West Windsor-Plainsboro Mathematics Expo

Saturday, October 26$^{\text{th}}$, 2019

Grade 6 Problem Set

Directions:

Solve the following problems to the best of your ability. If you do not understand a problem or cannot solve it, skip it or ask for a hint. If you cannot solve a problem even after receiving all the hints for that problem, wait until the 30 minute mark and ask a proctor for further help or the solution. Some problems may not have hints.

Calculators are not allowed for these problems. You may, however, discuss with the people around you after 30 minutes have passed. That being said, do not ruin a problem for somebody by giving them a solution before they have a chance to attempt the problem themselves.

For this test, there will be 20 questions, and you will have a time limit of 60 minutes in total, which will be split into 30 minutes of individual work and 30 minutes of collaborative work. This test is very long and you are not expected to be able to do all of the problems. We recommend picking a range of 10-15 problems to work on. Please note that this is not a competition, and your goal is to enjoy the problems and gain experience.

HAVE FUN!

By the way, if you finish this exceptionally early, you are most likely an exceptional student. Thus, here is a slightly harder problem that you may wish to solve:

CHALLENGE:

David is standing on the point (0,0) in a coordinate plane. Every minute, he travels a distance of 1 to a random lattice point. He stops moving if he stands on or travels past the lines $y = 2x - 2$ or $y = 2x + 3$. What is the expected time for him to stop moving?

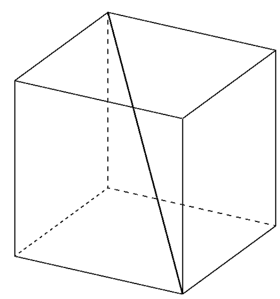
1. What is $2^{2^3} - 3^{2^2}$?

Solution:

$$2^{2^3} - 3^{2^2} = 2^8 - 3^4 = 256 - 81 = \boxed{175}$$

Hint: Simplify the values of the expressions in the exponents.

2. What is the length of the space diagonal of a cube with side length 3? (A space diagonal is a line connecting two vertices not on the same face)



Solution: Consider one of the face diagonals of the cube. It forms the hypotenuse of a right triangle with two of the sides, so it has a length of $\sqrt{3^2 + 3^2}$. The space diagonal is the hypotenuse of a right triangle with the face diagonal and a side as the legs, so it has length $\sqrt{\sqrt{3^2 + 3^2}^2 + 3^2}$, which is equal to $\sqrt{3^2 + 3^2 + 3^2}$ and simplifies to $\boxed{3\sqrt{3}}$.

Hint: Consider a face diagonal. What is its length and what type of triangle does it form with the space diagonal?

3. If 11 is the length of one of the legs of a right triangle and 61 is the length of the hypotenuse, what is the length of the other leg?

Solution: Let x be the unknown length. Using the Pythagorean Theorem, $11^2 + x^2 = 61^2$, or $x^2 = 3721 - 121 = 3600$. Notice $3600 = 36 \cdot 100$, so $x = 6 \cdot 10$, so the length of the leg is $\boxed{60}$.

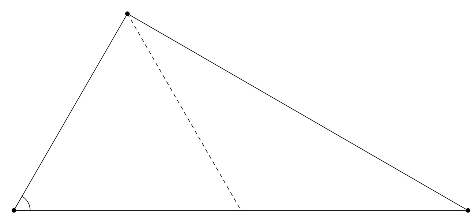
Hint: Use the Pythagorean Theorem.

4. What is the least positive integer n such that:

- $n \div 4$ has a remainder of 2
- $n \div 5$ has a remainder of 3
- $n \div 6$ has a remainder of 4
- $n \div 7$ has a remainder of 5?

> **Solution:** Consider the number $n+2$. It will be divisible by 4, 5, 6, and 7. This means that it must be divisible by $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$. The least positive number to satisfy this condition is obviously 420, so $n+2 = 420$. Therefore, $\boxed{n = 418}$.
>
> **Hint:** Consider the number $n+2$.

5. Let there exist $\triangle ABC$ with point D on $\overline{BC}$ such that $\overline{AD}$ is a median of the triangle. If $\angle B = 60\,\text{deg}$ and $AD = BD = 1$, find AC.



> **Solution:** Observe that $\triangle ABD$ is an equilateral triangle. Let E denote the foot of the altitude at A onto $\overline{BC}$. Then, ADE forms a 30-60-90 triangle, so $AE = \frac{\sqrt{3}}{2}$ and $ED = \frac{1}{2}$. Since $\overline{AD}$ is a median, $DC = BD = 1$ so $EC = ED + DC = \frac{1}{2} + 1 = \frac{3}{2}$. Using the Pythagorean Theorem on triangle AEC, $AC^2 = \frac{\sqrt{3}}{2}^2 + \frac{3}{2}^2 = \frac{3}{4} + \frac{9}{4} = 3$, and $\boxed{AC = \sqrt{3}}$.
>
> Alternatively, note that ADC is an isosceles triangle with $\angle C = 30\,\text{deg}$, so AEC is a 30-60-90 triangle.
> **Hint:** Draw the altitude from A onto BC.

6. David has 11 white socks and 9 black socks in a drawer. He takes two out at random, and then takes two more. What's the probability he first takes a pair of white socks, then a pair of black socks?

> **Solution:** The probability is the same as if David took the socks out one at a time. He has 20 socks in total, so the probability of first choosing a white sock is $\frac{11}{20}$. Now he has 10 white socks, and 19 socks in total, so the probability of choosing another white sock is $\frac{10}{19}$. Similarly, he now has a $\frac{9}{18}$ chance of choosing a black sock, and then a $\frac{8}{17}$ chance of choosing another black sock. The chance of David picking a pair of white socks and then a pair of black socks is therefore $\frac{11}{20} \cdot \frac{10}{19} \cdot \frac{9}{18} \cdot \frac{8}{17} = \boxed{\frac{22}{323}}$.
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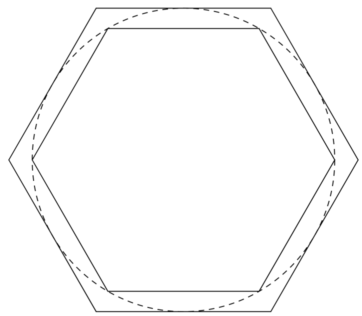
> **Hint:** Find the individual probabilities of each event.

7. What is the greatest power of 5 that divides 2019!?

Solution: The exponent of 5 in the prime factorization of a product is the sum of the exponents of 5 in the multiplicands. Therefore, each multiple of 5 (but not 25) should add 1 to the exponent of 2019!, each multiple of 25 (but not 125) should add 2, each multiple of 125 (but not 625) should add 3, and so on. Now consider the sum $\lfloor \frac{2019}{5} \rfloor + \lfloor \frac{2019}{25} \rfloor + \lfloor \frac{2019}{125} \rfloor + \ldots$ The first summand counts every multiple of 5 once, the second counts every multiple of 25 once, and so on. In total, this will add n for every number with a prime factorization that includes 5^n, as desired. Evaluating this sum, it is $\boxed{502}$.

Hint: How many times is a multiple of 5 counted? How many times is a multiple of 25 counted?

8. What is the ratio of the area of a regular hexagon inscribed in a circle to the area of a regular hexagon circumscribing the same circle?



Solution: Let the center of the circle be O and the radius equal r. The distance from O to a vertex P_1 of the inscribed hexagon is r. Letting P_2 denote a vertex of the circumscribed hexagon and Q denote the foot of an altitude at O onto a side adjacent to P_2, draw triangle OQP_2. $\angle QOP_2 = 30\,\text{deg}$, so $\triangle OQP_2$ is a 30-60-90 triangle. $OQ = r$ since it is a radius, so $OP_2 = \frac{2r}{\sqrt{3}}$. $\frac{OP_1}{OP_2} = \frac{\sqrt{3}}{2}$ represents the ratio of the lengths of the two hexagons, so the ratio of areas is $\boxed{\frac{3}{4}}$.

Hint: Use the center of the circle. What lengths of each hexagon are the same?

9. Let $i = \sqrt{-1}$. Find

$$\prod_{k=-4}^{4} (1 + ki)$$

Solution: For all x such that $1 \leq x \leq 4$, both $1 + xi$ and $1 - xi$ are terms in the desired product. Multiplying them together, $(1 + xi)(1 - xi) = 1 + x^2$. Therefore, the product is equivalent to $1(1 + 1^2)(1 + 2^2)(1 + 3^2)(1 + 4^2)$, which contains no imaginary term and can easily be simplified to $\boxed{1700}$.

Hint: Consider the number $(1 + ki)(1 - ki)$ for some integer k.

10. What is the probability that, given random numbers a and b such that $0 < a, b < 1$, there exists an obtuse triangle with side lengths a, b, and 1?

Solution: Since a and b are less than 1, then 1 is the longest side of the triangle. For it to be obtuse, it must meet the condition $a^2 + b^2 < 1$. Consider a square on the coordinate plane with points (0,0), (1,0), (1,1), and (0,1). If the x and y values of a coordinate represent a and b, respectively, then each point in the square represents a possible a and b. Thus, the desired probability is the fraction of the area of the square that satisfies $x^2 + y^2 < 1$. This is the equation for a unit circle with area π, so the probability is $\boxed{\frac{\pi}{4}}$.

Hint: How can the condition be represented as an algebraic inequality? How can probabilities with continuous variables be represented using geometry?

11. In the polynomial $2x^3 - 5x^2 + ax + b$, where a and b are integers, one of the roots is $2 + \sqrt{2}$. Find ab.

Solution: Because the polynomial coefficients are all integers, the conjugate of $2 + \sqrt{2}$, which is $2 - \sqrt{2}$, must also be a root. Therefore, $(x - 2 - \sqrt{2})(x - 2 + \sqrt{2}) = x^2 - 4x + 2$ is a factor of the polynomial. Since $x^2 - 4x + 2$ divides $2x^3 - 5x^2 + ax + b$, it also divides $2x^3 - 5x^2 + ax + b - 2x(x^2 - 4x + 2) = 3x^2 + (a - 4)x + b$. Clearly, $3x^2 + (a - 4)x + b$ must equal $3(x^2 - 4x + 2)$, so $a - 4 = 3(-4)$ and $b = 3(2)$. Solving for a and b, $a = -8$ and $b = 6$, so $\boxed{ab = -48}$.

Hint: What other root must the polynomial have?

12. What is the infinite sum $\frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \dots$?

Solution: Let S equal the infinite sum. $S - \frac{1}{2}S = \frac{1}{2} + (\frac{3}{4} - \frac{1}{4}) + (\frac{5}{8} - \frac{3}{8}) + (\frac{7}{16} - \frac{5}{16}) + \dots$. Simplifying the LHS and the expressions in the parentheses, $\frac{1}{2}S = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$. The RHS is the number $\frac{1}{2}$ added to a common infinite series which is equivalent to 1. Thus, $\frac{1}{2}S = \frac{3}{2}$, so $\boxed{S = 3}$.

Hint: What is half of the sum?

13. In terms of n, where $n \geq 4$, how many ways can David choose n balls such that:

- He only chooses red, green, or blue balls
- He chooses more than 3 red balls
- He chooses at most 4 green balls
- The number of blue balls he chooses is a multiple of 5?

(Assume balls of the same color are indistinguishable)

Solution: David has exactly 1 way to choose any number of combined blue and green balls. He has $n - 3$ ways to choose the number of red balls, so the total number of ways is $\boxed{n - 3}$.

Hint: How many ways can David choose a certain number of blue and green balls?

14. What are the last two digits of $3^5 \cdot 7^2 \cdot 11^2 \cdot 14$?

Solution: Begin by writing the prime factorization of this expression: $2 \cdot 3^5 \cdot 7^3 \cdot 11$. Consider this expression separately in both mod 4 and mod 25. There is only one factor of 2 that appears, so the expression is obviously equivalent to 2 (mod 4). $3^5 \equiv -7 \pmod{25}$ and $7^3 \equiv -7 \pmod{25}$. The entire expression is therefore equivalent to $2 \cdot (-7)^2 \cdot 11 \equiv 3 \pmod{25}$. By the Chinese Remainder Theorem, the expression is equivalent to 78 (mod 100), so the last two digits are $\boxed{78}$.

Hint: Take mod 4 and mod 25 separately.

15. What is the greatest integer that divides $2n^3(n-1)^2(n+1)(2n-1)$ for all positive integers n?

Solution: If n is even, then n^3 has at least three factors of 2, and if n is odd, both $(n-1)^2$ and $(n+1)$ have factors of 2, so, including the coefficient, there are always at least 4 factors of 2 in the expression, so the expression is always divisible by 16. If $n \equiv 0 \pmod 3$, then n^3 has at least three factors of 3. If $n \equiv 1 \pmod 3$, then $(n-1)^2$ has at least two factors of 3. If $n \equiv 2 \pmod 3$, then $(n+1)$ and $(2n-1)$ both contain at least one factor of three each, so the expression is always divisible by 48. If n is 2, then the expression is equal to 48, so the maximum number the expression is divisible by is $\boxed{48}$.

Hint: How many divisors of 2 must the expression have? How many divisors of 3? What other primes must divide the expression?

16. What is the minimum value of $f(x) = \frac{(x+2)^2}{x}$, given that x is positive?

Solution: Expanding the numerator, $f(x) = \frac{x^2+4x+4}{x}$. This can also be written as $f(x) = \frac{(x^2-4x+4)+8x}{x}$, or $f(x) = \frac{(x-2)^2}{x} + 8$. The first term is minimized when $x - 2 = 0$, or $x = 2$, which gives the minimum value of $f(x)$ as $\boxed{f(2) = 8}$.

Hint: What is the relationship between the desired expression and $\frac{(x-2)^2}{x}$?

17. In rectangle $ABCD$, point E lies on $\overline{AB}$ such that $\frac{AE}{ED} = 2$ and point F lies on the midpoint of $\overline{BC}$. Let points G and H lie on the intersections of $\overline{AC}$ with $\overline{DE}$ and $\overline{DF}$, respectively. If $AC = 15$, find GH.

Solution: Extend lines AB and DF until they meet at point I. Then $\triangle AIH \sim \triangle CDH$ with $AH = 2CH$, so $AH = 10$. Also, $\triangle AEG \sim \triangle CDG$ with $2CG = 3AG$, so $CG = 9$. Summing the lengths of AH and CG and subtracting the length of AC, $\boxed{GH = 4}$.

Hint: Extend lines AB and DF until they meet. What similar triangles exist?

18. Given that $f(x) = 2x^4 - 7x^3 + 13x^2 + 40x - 38$ has roots p, q, r, and s, find

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$$

Solution: The desired expression is equivalent to $\frac{pqr+pqs+pqrs+qrs}{pqrs}$. Using Vieta's formula, the numerator equals $-\frac{40}{2}$, while the denominator equals $\frac{-38}{2}$. Thus, the desired expression equals $\boxed{\frac{20}{19}}$.

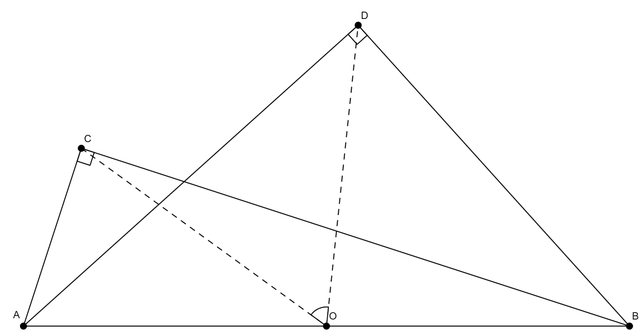
Hint: How can the expressions from Vieta's formula be manipulated into the desired expression?

19. What is the sum of the products of the elements of all the two-element subsets of $S = \{1, 2, 3, \dots, 20\}$?

Solution: Let x be the desired expression. Note that $x = 1 \cdot 2 + 1 \cdot 3 + ... + 1 \cdot 20 + 2 \cdot 3 + 2 \cdot 4 + \cdots + 2 \cdot 20 + ... + 18 \cdot 19 + 18 \cdot 20 + 19 \cdot 20$. Consider $(1 + 2 + 3 + ... + 20)^2$. Expanding this expression, it is equivalent to $(1^2 + 2^2 + 3^2 + ... + 20^2) + 2x$. The sum of the the first n integers is $\frac{n(n+1)}{2}$ and the sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$. From this, you have $2x = \frac{20^2(21)^2}{2^2} - \frac{20(21)(41)}{6}$. Tediously simplifying this expression gives $2x = 44100 - 2870 = 41230$, so $\boxed{x = 20615}$.

Hint: How does the expression $(1 + 2 + 3 + \cdots + 20)^2$ relate to the desired expression?

20. Let $\overline{AB}$ be a line segment with midpoint O. Let C and D be points on the same side of $\overline{AB}$ such that $\angle ACB$ and $\angle BDA$ are right angles and $\angle CAB = 72\,\text{deg}$ and $\angle DBA = 48\,\text{deg}$. What is $\angle COD$?



Solution: Notice that points A, B, C and D are concyclic with center O. The arc that $\angle CAB$ intercepts has length $144\,\text{deg}$ and the arc that DBA intercepts has length $96\,\text{deg}$. The sum of these two arc lengths is equal to the length of arcs AB and CD. The length of arc AB is $180\,\text{deg}$, meaning that the length of CD is $60\,\text{deg}$. Thus, $\boxed{\angle COD = 60\,\text{deg}}$.

Hint: Which points are concyclic?