The 1st Annual West Windsor-Plainsboro Mathematics Expo

Saturday, October 26$^{\text{th}}$, 2019

Grade 5 Problem Set

Directions:

Solve the following problems to the best of your ability. If you do not understand a problem or cannot solve it, skip it or ask for a hint. If you cannot solve a problem even after receiving all the hints for that problem, wait until the 30 minute mark and ask a proctor for further help or the solution. Some problems may not have hints.

Calculators are not allowed for these problems. You may, however, discuss with the people around you after 30 minutes have passed. That being said, do not ruin a problem for somebody by giving them a solution before they have a chance to attempt the problem themselves.

For this test, there will be 20 questions, and you will have a time limit of 60 minutes in total, which will be split into 30 minutes of individual work and 30 minutes of collaborative work. This test is very long and you are not expected to be able to do all of the problems. We recommend picking a range of 10-15 problems to work on. Please note that this is not a competition, and your goal is to enjoy the problems and gain experience.

HAVE FUN!

By the way, if you finish this exceptionally early, you are most likely an exceptional student. Thus, here is a slightly harder problem that you may wish to solve:

CHALLENGE:

You are given an 8x8 chessboard, 21 3x1 dominoes, and one 1x1 domino. Is it possible to tile the board with the dominoes? Note that dominoes cannot overlap.

1. What are the next two numbers in the pattern? $1, 4, 16, 64, \cdots$

> **Solution:** The pattern is each number is four times the previous term. Thus, the next two numbers are $\boxed{256, 1024}$.
>
> **Hint:** Try to divide numbers next to each other.

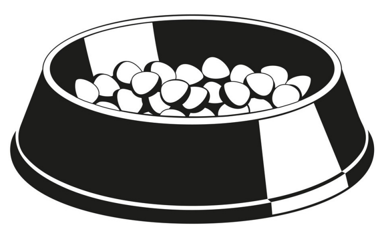
2. What is 90% of 110% of 100?

> **Solution:** $100 \cdot \frac{110}{100} \frac{90}{100} = \boxed{99}$.
>
> **Hint:** It is not 100.

3. 100 students filled out a survey. 60 said they own a dog, 50 said they own a cat, and 10 said they owned neither. How many owned both a cat and a dog?

> **Solution:** We first disregard the 10 people who own neither, leaving 90 students remaining. By the principle of inclusion and exclusion, we get $60 + 50 - x = 90$. Solving for x yields $x = \boxed{20}$.
>
> **Hint:** If I own a dog and a cat, then I am counted separately for owning a dog, owning a cat, and for owning both. venn diagrams

4. If 2 dogs eat 5 cans of dog food, how many cans of dog food do 7 dogs eat?



> **Solution:** We see that each dog eats $\frac{5}{2}$ cans of dog food. Thus, 7 dogs eat $7 \cdot \frac{5}{2} = \boxed{\frac{35}{2}}$.
>
> **Hint:** How many cans of dog food does one dog eat?

5. Jack is going to make an order at a restaurant. He has 3 options for his meal, 2 options for his drink, and 2 options for his appetizer. If an order consists of exactly one meal, one drink, and one appetizer, how many orders can Jack choose from?

> **Solution:** By the counting principle, there are $3 \cdot 2 \cdot 2 = \boxed{12}$ orders.
>
> **Hint:** What if an order only consists of one meal and one drink?

6. What is the area of the triangle defined by the coordinates (0,0), (6,4), and (4,6)?

Solution: After graphing these points, consider the rectangle defined by the points (0,0), (0,6), (6,6), and (6,0). Subtracting the 3 excess triangles from the rectangle gives $6 \cdot 6 - \frac{1}{2} \cdot 6 \cdot 4 - \frac{1}{2} \cdot 4 \cdot 6 - \frac{1}{2} \cdot 2 \cdot 2 = 36 - 12 - 12 - 2 = \boxed{10}$.

Hint: Overestimate the area and then correct it with subtraction areas.

7. What is the area of a triangle with side lengths 333, 444, and 555?

Solution: Note that this is a 3-4-5 triangle scaled by 111. Thus, the sides of length 3334 and 444 are perpendicular, giving an area of $\frac{1}{2} \cdot 333 \cdot 444 = \boxed{73926}$.

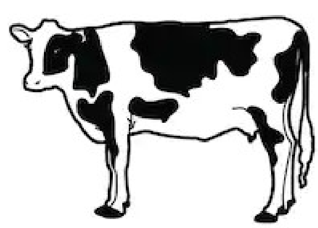
Hint: Is there something special about this triangle?

8. What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Solution: $\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = \boxed{1}$.

Hint: Let the sum equal x. What is $2 \cdot x$?

9. Farmer John's cows have been oddly quiet. To solve this, he is giving his 4 cows some bells. In how many ways can he distribute 3 identical bells to the 4 (not identical) cows?



Solution: We can represent the cows with 3 dividers(///) and the 3 bells as 3 identical circles(ooo). Each space in between dividers represents a cow (including before the first and after the last) so all four cows are shown. Finding the number of ways to order this sequence gives us the total numbers of ways to distribute the bells. Thus, we get $\frac{6!}{3!3!} = \boxed{20}$ ways.

Hint: Can you represent this problem using letters?

10. Quadrilateral ABCD is a square with side length 4. Points M and N are the midpoints of AB and CD respectively. Point P is the intersection of AN and MD and point Q is the intersection of MC and BN. What is the area of quadrilateral MPNQ?

Solution: We first note that the quadrilateral MPNQ is composed of the two congruent triangles MPQ and NPQ. Both of these have a height of $\frac{4}{2} = 2$. By symmetry, they have a base of $2 \cdot \frac{4}{4} = 2$. Thus, the total area is $2 \cdot \frac{1}{2} \cdot 2 \cdot 2 = \boxed{4}$

Hint: Draw a picture and look for symmetry.

11. For what values of x is $x^3 - x = 0$?

Solution: Factoring yields $x(x^2 - 1) = x(x + 1)(x - 1) = 0$, so $x = \boxed{0, 1, -1}$.

Hint: Can you factor this equation?

12. Before plums are dried to become prunes, they are 82% water. After they dried, they are only 10% water. If only water is lost in the drying process, how many pounds of prunes can be made from 100 pounds of plums?

Solution: Since only water is lost in the drying process, the same amount of plum is in the plums both before and after the drying process. We see that we have $100 \cdot (1 - \frac{82}{100}) = 18$ pounds of plum initially, implying that we have 18 pounds after the drying process. Thus, we see that the 18 pounds are the 90% of plum at the end, meaning that there are $18 \cdot \frac{100}{90} = \boxed{20}$ pounds of prunes at the end.

Hint: What doesn't change when a plum becomes a prune?

13. Jim is making a model of a building that is shaped like a rectangular prism. He decides on a ratio of 1 foot to 1 inch for his model. If his model is 2 inches x 3 inches x 4 inches, how many times larger is the volume of the building than the model? (Note: there are 12 inches in a foot)

Solution: For each dimension, the volume scales by 12. Thus, the final volume is $12^3 = \boxed{1728}$ times larger.

Hint: Think about the problem in 2 dimensions first.

14. How many ways are there to get from (0,0) to (4,4) if you can only move 1 unit upward or to the right at a time?

> **Solution:** A path contains 4 steps to the left and 4 steps up. The number of ways to order this is $\frac{8!}{4!4!} = \boxed{70}$
>
> **Hint:** How many times do you need to go left? What about up? How many ways can you order those steps?

15. A circle is inscribed in an equilateral triangle, which is in turn inscribed in another circle. What is the ratio between the areas of the two circles?

> **Solution:** Note that both centers share the same center. Drawing the altitudes of the triangle split each length into segments with lengths equal to the incircle's radius and the circumcircle's radius. Since these altitudes are also medians, they divide each other in lengths of ratio 2:1. Using the fact that the height is $\frac{\sqrt{3}}{2}$ (this can be derived from the area), we get the smaller radius to be $\frac{\sqrt{3}}{6}$ and the larger one to be $\frac{\sqrt{3}}{3}$. We can see that the areas of the two circles are $\pi\frac{3}{36} = \frac{\pi}{12}$ and $\pi\frac{3}{9} = \frac{\pi}{3}$. We can now see that the ratio between the areas is $\boxed{1:4}$.
>
> **Hint:** Draw a large and accurate diagram.

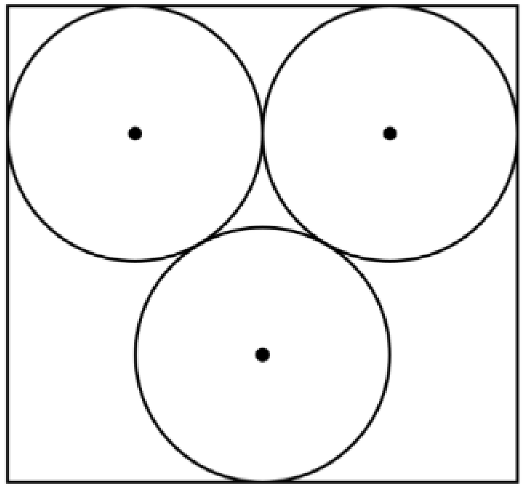
16. What is $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{99\cdot 100}$

> **Solution:** We can set up the telescoping sum using $\frac{1}{(n)(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ to get $\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100} = 1 - \frac{1}{100} = \boxed{\frac{99}{100}}$
>
> **Hint:** What is the sum of the first 2 terms? First 3? First 4?

17. Find the area of a regular octagon with side length 1.

> **Solution:** We can divide the octagon into 9 pieces by connecting each vertex with the one directly across it. This gives us a square, 4 rectangles, and 4 triangles. The area of the square is $1 \cdot 1 = 1$. The area of a rectangle is $1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$. The area of a triangle is $\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}$. Thus, the total area is $1 \cdot 1 + 4 \cdot \frac{\sqrt{2}}{2} + 4 \cdot \frac{1}{4} = \boxed{2 + 2\sqrt{2}}$
>
> **Hint:** Can you divide an octagon intro simpler shapes?

18. Given three tangent circles of radius 1, what is the area of the region between them?



Solution:

We first connect the centers of the three circles to form an equilateral triangle of side length 2. Thus, the desired area is simply the triangle's area minus the areas of the three sectors. This gives us $A = \frac{2^2\sqrt{3}}{4} - 3 \cdot \frac{1}{6} \cdot \pi \cdot 1^2 = \boxed{\sqrt{3} - \frac{\pi}{2}}$

Hint: Can you make a triangle?

19. Let X be a positive integer less than 300. When X is divided by 2, the remainder is 1. When X is divided by 3, the remainder is 2. When X is divided by 5, the remainder is 4. When X is divided by 7, the remainder is 6. Find X.

Solution: Note that all of these remainders are 1 less than the divisor. Thus, X is simply $2 \cdot 3 \cdot 5 \cdot 7 - 1 = 210 - 1 = \boxed{209}$

Hint: Find the difference between the remainders and the divisors.

20. What is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Solution: Consider the sum in pieces of size 2^n (1, 2, 4, 8, etc.). For the piece of size 2^n, each of the fractions will be greater than or equal to $\frac{1}{2^{n+1}}$, as the piece starts with $\frac{1}{2^n}$ and ends with $\frac{1}{2^{n+1}-1}$. Since these are all of these fractions are greater than or equal to $\frac{1}{2^{n+1}}$, we know that their sum is greater than $2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$. We can repeat this process indefinitely, implying that the sum is infinite, or $\boxed{\infty}$.

Hint: Break up the sum by powers of 2.