The 1st Annual West Windsor-Plainsboro Mathematics Expo

Saturday, October 26$^{\text{th}}$, 2019

Grade 7 Problem Set

Directions:

Solve the following problems to the best of your ability. If you do not understand a problem or cannot solve it, skip it or ask for a hint. If you cannot solve a problem even after receiving all the hints for that problem, wait until the 30 minute mark and ask a proctor for further help or the solution. Some problems may not have hints.

Calculators are not allowed for these problems. You may, however, discuss with the people around you after 30 minutes have passed. That being said, do not ruin a problem for somebody by giving them a solution before they have a chance to attempt the problem themselves.

For this test, there will be 20 questions, and you will have a time limit of 60 minutes in total, which will be split into 30 minutes of individual work and 30 minutes of collaborative work. This test is very long and you are not expected to be able to do all of the problems. We recommend picking a range of 10-15 problems to work on. Please note that this is not a competition, and your goal is to enjoy the problems and gain experience.

HAVE FUN!

By the way, if you finish this exceptionally early, you are most likely an exceptional student. Thus, here is a slightly harder problem that you may wish to solve:

CHALLENGE:

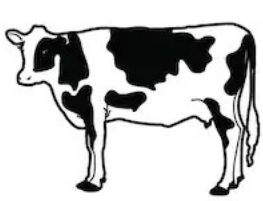
Let a and b be positive integers such that $(1 + ab)|(a^2 + b^2)$. Show that $\frac{(a^2+b^2)}{(1+ab)}$ must be a perfect square.

If you think it is too hard, don't worry, because it is the toughest problem on IMO :)

1. Evaluate $(2^{(2^2)})) - (1^{(1^1)})$

2. Find the probability of getting a sum of 8 after rolling two fair 6-sided dice.

3. One day, Bessie the cow decides to wander in her farm. First, she walks west for 15 miles, north 12 miles, and east 2 miles. Find the distance between her endpoint and starting point.



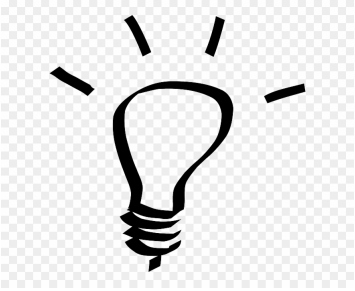
4. Find the unit digit of $2019^{2019} - 2019!$.

5. Find the last 2 digits of $2019! - 25^{2019}$.

6. What is the area of hexagon $\overline{ABCDEF}$ with A=(1, 1), B=(4, 2), C=(9, 3), D=(16, 4), E=(25, 5), F=(36, 6).

7. Let $f(x) = 1 + \frac{1}{x}$. Find $\prod_{i=1}^{2019} f(i)$. Note: This is simply asking you to find $f(1) \cdot f(2) \cdot f(3) \ldots \cdot f(2019)$.

8. One day, Bessie the cow encountered 2020 rooms. In each room, there is one light bulb, and all the light bulbs are initially off. Since Farmer John is not supervising her, she decides to play around with the rooms. Specifically, she wants to turn on some of the light bulbs. She visits all the rooms, and she has $\frac{1}{2}$ of chance to turn on each light bulb. But she is very afraid of darkness, so at the end of the process, it is given at least one light bulb that is turned on. Find the expected number of light bulbs that is on at the end of the process.

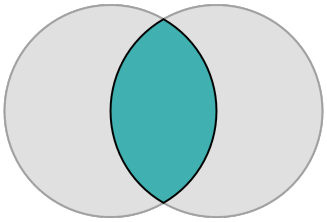


9. Let f(x) be a monic 4th degree polynomial such that f(1) = 2, f(2) = 5, f(3) = 10, f(4) = 17. Find f(5).

10. Find the number of ways to arrange 3 reds identical beads, 3 blue identical beads, and 3 green identical beads on a necklace. Given that 2 arrangements are considered different if and only if one cannot obtain the other by rotation or reflection.

11. Let $t(x)$ denotes the number of divisors of x. Find the sum of a less than 60 such that $t(a) - t(a-1)$ is a positive odd integer.

12. Let w_1 and w_2 be two circles that intersect at P and Q. Let A, B be on w_1, and C, D be on w_2 such that $\overline{AC}$ passes through P and $\overline{BD}$ passes through P. Given $\overline{QA} = 4$, $\overline{QC} = 6$, $\overline{QB} = 2$, find $\overline{QD}$.



13. Let a_n and b_n be a sequence of real numbers, and $a_0 = 1$ and $b_0 = 1$. In addition for $k > 0$, $a_k = a_{k-1}^2 - b_{k-1}^2$, and $b_k = 2a_{k-1}b_{k-1}$. Find $(a_{2019})^2 + (b_{2019})^2$.

14. Find the first positive integer n such that $3^n \equiv 2019^n \pmod{343}$.

15. Let P be a point in $\triangle$ ABC. Given that AB=AC, $\angle$ BAC = 102°, $\angle$ PBA=9°, $\angle$ PAB=21°. Find the degree of $\angle$ APC.

16. Tossing dice has always been Elsie's favorite activity. She is so addicted that she wouldn't give up until she tosses all numbers from 1 through 3 consecutively in that order at least once. Find the probability that Elsie finishes tossing her dice using exactly 9 tosses?

17. Find the following sum: $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m}{3^{n+m} \cdot (m+n)}$

18. Find the maximum number of different values that can be obtained from the expression a/b/c/d/e/f using 5 pairs of parentheses for some real non-zero a, b, c, d, e, f. Every pair of parentheses must be non-trivial. i. e, "(a)" or "((a/b))" are not allowed.

19. Find the minimum real value of $\sqrt{x^2 + 20x + 1000} + \sqrt{x^2 - 8x + 272}$

20. Let w_1 and w_2 be two circles, and let l be the external common tangent line which tangents to w_1 and w_2 at A and B, respectively. Let P and Q be the two intersections of the circles, and P is closer to l than Q. Let $\overline{AC}$ be parallel to $\overline{BQ}$ and C is on w_1. Let $\overline{BD}$ be parallel to $\overline{AQ}$ and D is on w_2. Given that $\angle$PAB = 15°, and $\angle$PBA = 30°. Find $\overline{CP}/\overline{DP}$.