The 1st Annual West Windsor-Plainsboro Mathematics Expo

Saturday, October 26^{th}, 2019

Grade 6 Problem Set

Directions:

Solve the following problems to the best of your ability. If you do not understand a problem or cannot solve it, skip it or ask for a hint. If you cannot solve a problem even after receiving all the hints for that problem, wait until the 30 minute mark and ask a proctor for further help or the solution. Some problems may not have hints.

Calculators are not allowed for these problems. You may, however, discuss with the people around you after 30 minutes have passed. That being said, do not ruin a problem for somebody by giving them a solution before they have a chance to attempt the problem themselves.

For this test, there will be 20 questions, and you will have a time limit of 60 minutes in total, which will be split into 30 minutes of individual work and 30 minutes of collaborative work. This test is very long and you are not expected to be able to do all of the problems. We recommend picking a range of 10-15 problems to work on. Please note that this is not a competition, and your goal is to enjoy the problems and gain experience.

HAVE FUN!

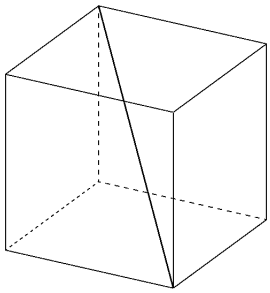
By the way, if you finish this exceptionally early, you are most likely an exceptional student. Thus, here is a slightly harder problem that you may wish to solve:

CHALLENGE:

David is standing on the point (0,0) in a coordinate plane. Every minute, he travels a distance of 1 to a random lattice point. He stops moving if he stands on or travels past the lines $y = 2x - 2$ or $y = 2x + 3$. What is the expected time for him to stop moving?

1. What is $2^{2^3} - 3^{2^2}$?

2. What is the length of the space diagonal of a cube with side length 3? (A space diagonal is a line connecting two vertices not on the same face)

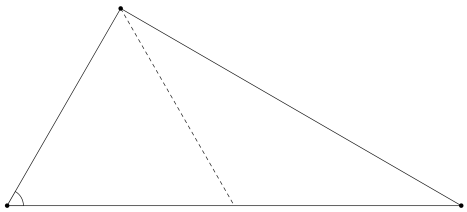


3. If 11 is the length of one of the legs of a right triangle and 61 is the length of the hypotenuse, what is the length of the other leg?

4. What is the least positive integer n such that:

   - $n \div 4$ has a remainder of 2
   - $n \div 5$ has a remainder of 3
   - $n \div 6$ has a remainder of 4
   - $n \div 7$ has a remainder of 5?

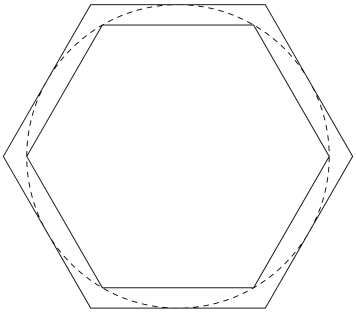
5. Let there exist $\triangle ABC$ with point D on $\overline{BC}$ such that $\overline{AD}$ is a median of the triangle. If $\angle B = 60\,\text{deg}$ and $AD = BD = 1$, find AC.



6. David has 11 white socks and 9 black socks in a drawer. He takes two out at random, and then takes two more. What's the probability he first takes a pair of white socks, then a pair of black socks?

7. What is the greatest power of 5 that divides 2019!?

8. What is the ratio of the area of a regular hexagon inscribed in a circle to the area of a regular hexagon circumscribing the same circle?



9. Let $i = \sqrt{-1}$. Find

$$\prod_{k=-4}^{4} (1 + ki)$$

10. What is the probability that, given random numbers a and b such that $0 < a, b < 1$, there exists an obtuse triangle with side lengths a, b, and 1?

11. In the polynomial $2x^3 - 5x^2 + ax + b$, where a and b are integers, one of the roots is $2 + \sqrt{2}$. Find ab.

12. What is the infinite sum $\frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \ldots$?

13. In terms of n, where $n \geq 4$, how many ways can David choose n balls such that:

    - He only chooses red, green, or blue balls
    - He chooses more than 3 red balls
    - He chooses at most 4 green balls
    - The number of blue balls he chooses is a multiple of 5?

    (Assume balls of the same color are indistinguishable)

14. What are the last two digits of $3^5 \cdot 7^2 \cdot 11^2 \cdot 14$?

15. What is the greatest integer that divides $2n^3(n-1)^2(n+1)(2n-1)$ for all positive integers n?

16. What is the minimum value of $f(x) = \frac{(x+2)^2}{x}$, given that x is positive?

17. In rectangle $ABCD$, point E lies on $\overline{AB}$ such that $\frac{AE}{ED} = 2$ and point F lies on the midpoint of $\overline{BC}$. Let points G and H lie on the intersections of $\overline{AC}$ with $\overline{DE}$ and $\overline{DF}$, respectively. If $AC = 15$, find GH.

18. Given that $f(x) = 2x^4 - 7x^3 + 13x^2 + 40x - 38$ has roots p, q, r, and s, find

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s}$$

19. What is the sum of the products of the elements of all the two-element subsets of $S = \{1, 2, 3, \ldots, 20\}$?

20. Let $\overline{AB}$ be a line segment with midpoint O. Let C and D be points on the same side of $\overline{AB}$ such that $\angle ACB$ and $\angle BDA$ are right angles and $\angle CAB = 72\,\text{deg}$ and $\angle DBA = 48\,\text{deg}$. What is $\angle COD$?